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KLEINIAN CIRCLE PACKINGS

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RECENTLY there has been some interest in finitely generated Kleinian groups whose limit set is a circle packing [6, 7, 9]. These correspond to hyperbolic three manifolds with totally geodesic boundary. It has been asked when these limit sets have circle packing exponent equal to their Hausdorff dimension. The purpose of this paper is to answer this question for a large class of groups. The circle packing exponent is of interest because it is easy to calculate, see [6, 9], but the Hausdorff dimension is a more useful theoretical invariant, see [11, 3].

It is known that for the Apollonian packing the circle packing exponent and the Hausdorff dimension are equal [5, 12]. Moreover, for an arbitrary sphere packing of a compact subset in \mathbb{R}^n the Hausdorff dimension is at most the sphere packing exponent [8]. There are examples of circle packings where the circle packing exponent is strictly greater than the Hausdorff dimension [4] but these examples are not invariant under a group.

Let K be a compact subset of \mathbb{C} . Remove from K a countable union of disjoint open discs whose total area equals that of K . What remains is a circle packing Π of K . By a circle in this packing we will always mean the boundary of one of these removed discs (including the boundary of K if K is a compact disc). Let r denote the radius of an arbitrary circle in Π . The circle packing exponent (see [8, 5]) of Π is defined to be

$$e = \inf \left\{ t: \sum_{\text{circles in } \Pi} r^t < \infty \right\} = \sup \left\{ t: \sum_{\text{circles in } \Pi} r^t = \infty \right\}.$$

It is clear that $e \leq 2$ as when $t = 2$ the sum is finite (in fact it is $\text{area}(K)/\pi$).

A Kleinian group G is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. The limit set $\Lambda(G)$ of G is the subset of the Riemann sphere where G does not act discontinuously. The complement of $\Lambda(G)$ in the Riemann sphere is called the ordinary set $\Omega(G)$ of G . A Kleinian group G is geometrically finite if it has a finite sided fundamental polyhedron (see [2, 11]). The subgroup of a geometrically finite group G stabilising the fixed point of a parabolic element of G is either a finite extension of a cyclic group or contains a subgroup isomorphic to \mathbb{Z}^2 . Moreover, when this stabiliser is a finite extension of a cyclic group, there exist disjoint open discs B_1 and B_2 contained in $\Omega(G)$ and tangent at the parabolic fixed point [2].

The circle packings we consider here will all be the limit sets $\Lambda(G)$ of finitely generated Kleinian groups G and will be of two types (see Figs 1–4). Groups G of the first type have the point ∞ in their ordinary set (Figs 2 and 4). That is, $\Lambda(G)$ is a circle packing of a compact disc K in \mathbb{C} and the subgroup G_∞ of G stabilising ∞ is finite. Groups of the second type will have ∞ a parabolic fixed point (Figs 1 and 3). Again, we denote the subgroup of

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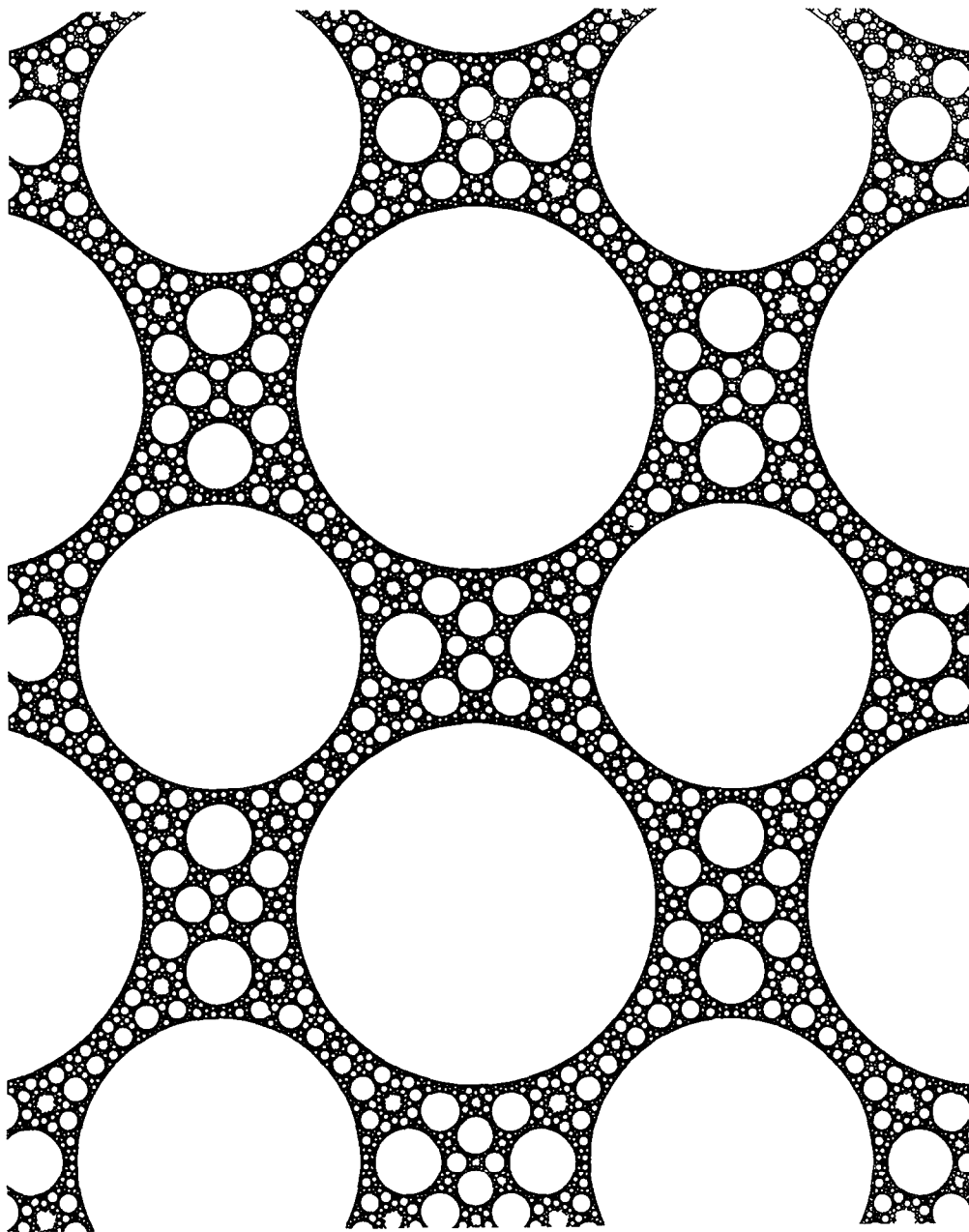


Fig. 1. The limit set of group G for which G_∞ contains a subgroup isomorphic to \mathbb{Z}^2 .

G stabilising ∞ by G_∞ . Of course, we exclude groups whose limit set is a single circle, that is finite extensions of Fuchsian groups.

A circle will be called infinite if it is the union of a straight line and the point ∞ of the Riemann sphere. When G_∞ is a finite extension of a cyclic group then there are necessarily exactly two infinite circles in $\Lambda(G)$ and $\Lambda(G)$ is contained in the strip between them (Fig. 3). Otherwise all circles are finite. The orbit of a given finite circle under G_∞ consists of circles all with the same radius. When ∞ is a parabolic fixed point there are infinitely many of these and so for all t the series used to define e is infinite. In order to make a sensible

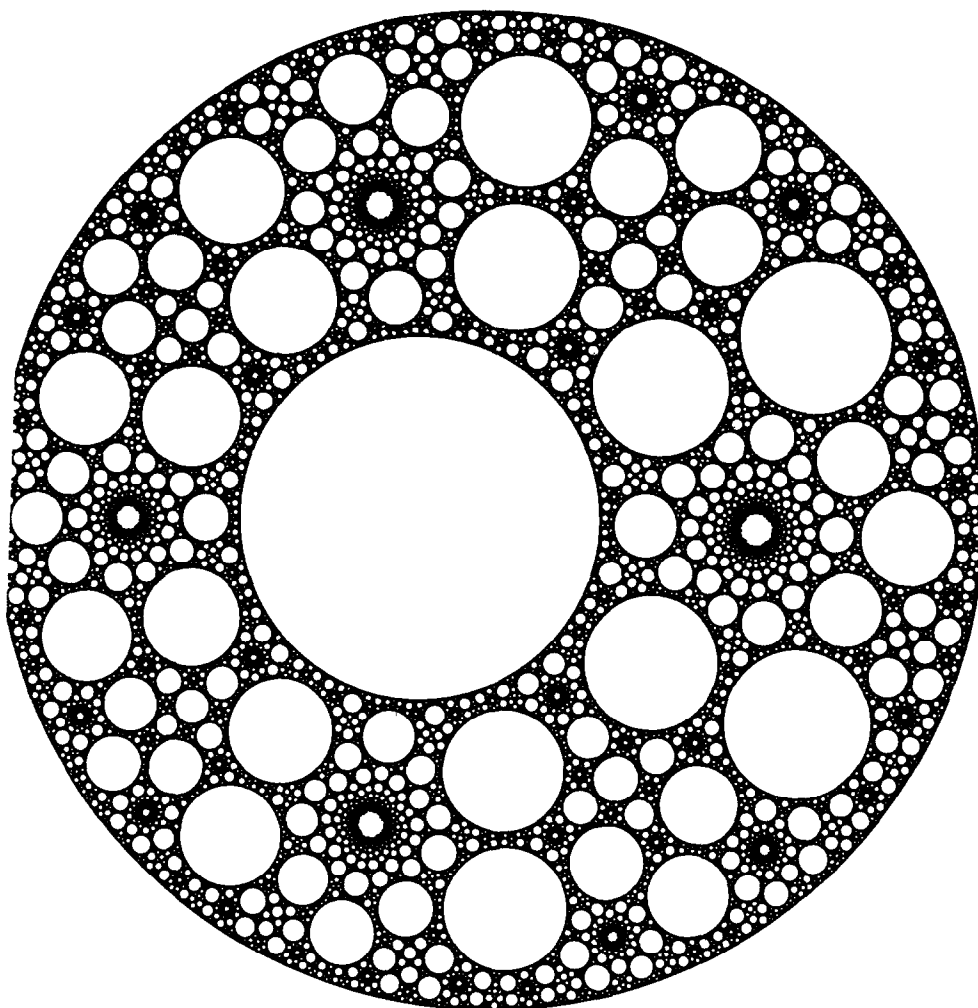


Fig. 2. The limit set of a conjugate of the previous group with infinity in the ordinary set.

definition of the packing exponent we change the definition of e by summing over all G_∞ -orbits of finite circles. This agrees with the previous definition for groups of the first type. That is, we define

$$e = \inf \left\{ t: \sum_{G_\infty\text{-orbits of finite circles in } \Lambda(G)} r^t < \infty \right\}.$$

It is easy to see that $e \leq 2$ as when $t = 2$ the sum is proportional to the area of that part of the packing contained in a fundamental domain for G_∞ , and so is finite.

The object of this paper is to prove the following theorem.

THEOREM 1. *Let G be a finitely generated Kleinian group whose limit set $\Lambda(G)$ is a circle packing. Suppose that ∞ is either in the ordinary set of G or is a parabolic fixed point. Then the circle packing exponent e of $\Lambda(G)$ equals the Hausdorff dimension d of $\Lambda(G)$.*

A result of Larman [8] states that for any circle packing Π of a compact subset K of \mathbb{C} we have $e \geq d$. Recently, Bishop and Jones [3] have proved that $d = 2$ when G is

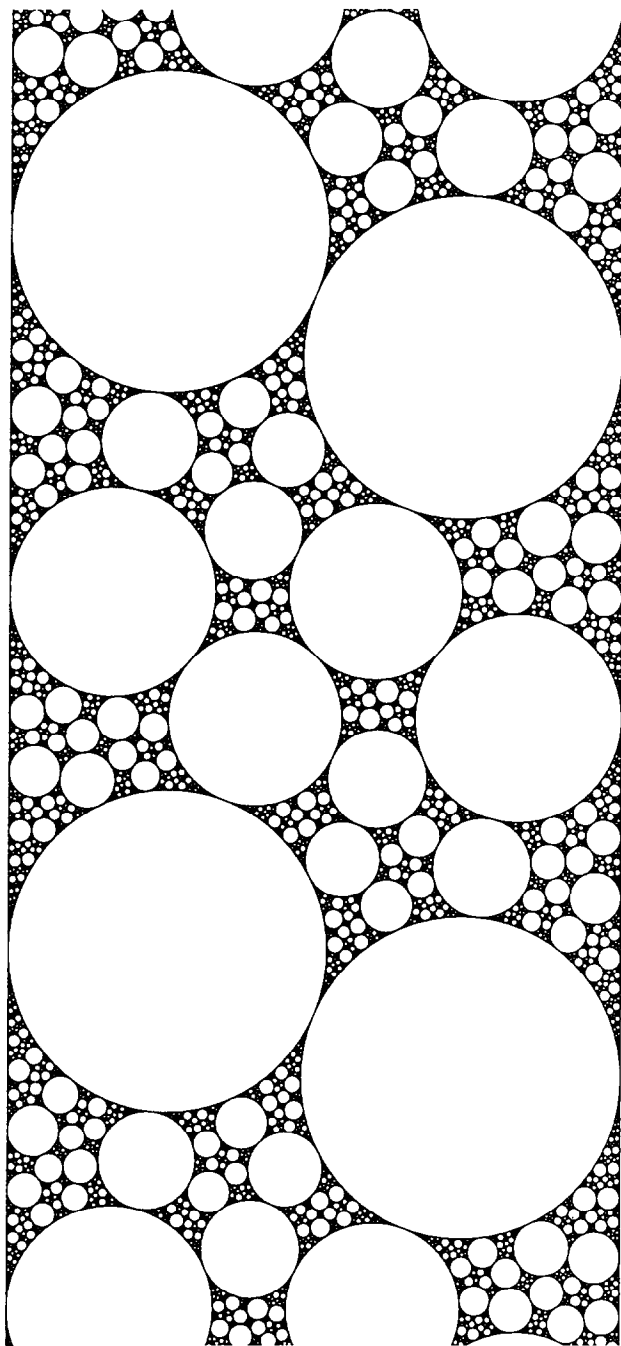


Fig. 3. The limit set of group G for which G_∞ is a finite extension of a cyclic group.

geometrically infinite, in which case $2 = d \leq e \leq 2$ and so $d = e = 2$. It remains to prove that $e \leq d$ when G is geometrically finite. We begin with a simple lemma.

LEMMA 2. Let $g \in \mathbf{PSL}(2, \mathbb{C})$ be given by $g(z) = (az + b)/(cz + d)$ where $ad - bc = 1$.
(i) The image of the unit circle under g is a circle centred at $z_1(g)$ with radius $r_1(g)$ where

$$z_1(g) = \frac{a\bar{c} - b\bar{d}}{|c|^2 - |d|^2}, \qquad r_1(g) = \frac{1}{||c|^2 - |d|^2|}.$$

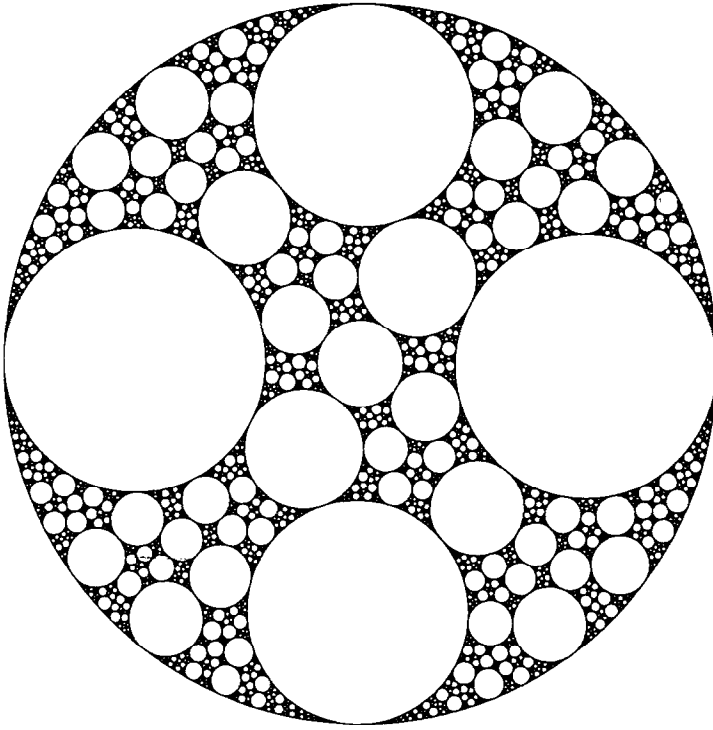


Fig. 4. The limit set of a conjugate of the previous group with infinity in the ordinary set.

If $|c| = |d|$ then $r_1(g)$ is infinite and $z_1(g)$ is undefined.

(ii) The image of the extended real line under g is a circle centred at $z_2(g)$ with radius $r_2(g)$ where

$$z_2(g) = \frac{a\bar{d} - b\bar{c}}{c\bar{d} - d\bar{c}}, \quad r_2(g) = \frac{1}{|c\bar{d} - d\bar{c}|}.$$

If $c\bar{d} = d\bar{c}$ then $r_2(g)$ is infinite and $z_2(g)$ is undefined.

Proof. Once these formulae are known they are not hard to verify: for any $e^{i\theta}$ on the unit circle

$$|g(e^{i\theta}) - z_1(g)| = \left| \frac{ae^{i\theta} + b}{ce^{i\theta} + d} - \frac{a\bar{c} - b\bar{d}}{|c|^2 - |d|^2} \right| = \frac{|ad - bc| | -\bar{c} - \bar{d}e^{i\theta} |}{||c|^2 - |d|^2| |ce^{i\theta} + d|} = \frac{1}{||c|^2 - |d|^2|} = r_1(g).$$

For any $x \in \mathbb{R}$,

$$|g(x) - z_2(g)| = \left| \frac{ax + b}{cx + d} - \frac{a\bar{d} - b\bar{c}}{c\bar{d} - d\bar{c}} \right| = \frac{|ad - bc| | -\bar{c}x - \bar{d} |}{|c\bar{d} - d\bar{c}| |cx + d|} = \frac{1}{|c\bar{d} - d\bar{c}|} = r_2(g).$$

Similarly $|g(\infty) - z_2(g)| = r_2(g)$. □

In the next proposition we compare the series used in the definition of the circle packing exponent to a series considered by Beardon [1]. There he shows that this sum has the same exponent of convergence as the Poincaré series. This comparison is the main ingredient in the proof of Theorem 1.

PROPOSITION 3. *Suppose that G is geometrically finite and satisfies the conditions of Theorem 1. Let G_∞ denote the stabiliser of ∞ . Then there exists a constant C depending only on G so that for $t \leq 2$,*

$$\sum_{G_\infty\text{-orbits of finite circles in } \Lambda(G)} r^t \leq C \sum_{g \in G_\infty \backslash G/G_\infty - \{I\}} |c|^{-2t}$$

where $g(z) = (az + b)/(cz + d)$ with $ad - bc = 1$.

Proof. There are only a finite number of G -orbits of circles in $\Lambda(G)$, corresponding to the ends of the hyperbolic three manifold \mathbb{H}^3/G . We treat each orbit separately and then combine the results at the end.

We begin by considering G -orbits consisting of finite circles. Consider the images under G of one circle γ . Conjugating by a Euclidean similarity we may assume that γ is the unit circle. This conjugation scales the radius of every circle uniformly and so changes the sums we are interested in by at most a constant multiple. Let H be the subgroup of G sending γ to itself. Then H is the fundamental group of the end of \mathbb{H}^3/G corresponding to γ . In order to count each circle exactly once we sum over G/H but in order to count each G_∞ -orbit of circles once we need to sum over $G_\infty \backslash G/H$. We now show how to index these cosets.

Let U be any subset of $\hat{\mathbb{C}}$. Define $P(U) = \{z \in U: \text{there exists } g \in G \text{ with } g(z) = \infty\}$. We claim that the cosets $G_\infty \backslash G/H$ may be indexed by the points in $P(D)$ where D is a fundamental domain for the action H on $\Omega(H) = \hat{\mathbb{C}} - \gamma$. As the orbit of γ was supposed to consist only of finite circles we see that $g^{-1}(\infty)$ cannot lie on γ . Thus, all preimages of ∞ lie in $\Omega(H)$ and may be carried to D by a unique element of H . So a representative from each coset G/H is uniquely specified by choosing $g \in G$ so that $g^{-1}(\infty)$ lies in a fundamental domain for H . Our coset may then be written as gH . We now investigate which cosets correspond to the same point $g^{-1}(\infty)$. Observe that g_1H and g_2H are two such cosets if and only if $g_1g_2^{-1} \in G_\infty$. That is, g_1H and g_2H are in the same coset of $G_\infty \backslash G/H$. So we may index the cosets $G_\infty \backslash G/H$ by considering points $g^{-1}(\infty)$ in a given fundamental domain for H , that is by points of $P(D)$ as claimed. This gives

$$\sum_{g \in G_\infty \backslash G/H} r_1(g)^t = \sum_{g \in G_\infty \backslash G/H} \frac{1}{||c|^2 - |d|^2|^t} = \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{||c|^2 - |d|^2|^t}.$$

We choose a fundamental domain D for H that is the union of two hyperbolic polygons, one in each of the discs bounded by γ . These polygons have finite hyperbolic area in each of these discs and finitely many vertices on γ (Ahlfors' finiteness theorem). As γ is contained in a fundamental domain for G_∞ we may choose D also lying in a fundamental domain for G_∞ , call this fundamental domain E . The vertices of D on γ correspond to conjugacy classes of parabolics in H . As G is geometrically finite each of these vertices is doubly cusped (see [2]). This means that for each vertex p of D on γ there are disjoint discs B_1 and B_2 tangent to γ at p so that the intersection of B_j with D contains only finitely many G -images of ∞ . Thus, there exists $\varepsilon > 0$ so that all G -images of ∞ in D are at least a distance ε from γ . If $g(z) = (az + b)/(cz + d)$ with $ad - bc = 1$ this is $||-d/c| - 1| \geq \varepsilon$. A simple calculation shows that this implies $||-d/c|^2 - 1| \geq \varepsilon$. So we have

$$\begin{aligned} \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{||c|^2 - |d|^2|^t} &\leq \frac{1}{\varepsilon^t} \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{|c|^{2t}} \\ &\leq \frac{1}{\varepsilon^{2t}} \sum_{g^{-1}(\infty) \in P(E)} \frac{1}{|c|^{2t}} \end{aligned}$$

as $P(D)$ is a subset of $P(E)$ and $t \leq 2$. Now the points of $P(E)$ index the cosets $G_\infty \backslash G/G_\infty - \{I\}$ in the same way that the points of $P(D)$ index the cosets $G_\infty \backslash G/H$. Thus,

$$\sum_{g^{-1}(\infty) \in P(E)} \frac{1}{|c|^{2t}} = \sum_{g \in G_\infty \backslash G/G_\infty - \{I\}} \frac{1}{|c|^{2t}}.$$

This proves the result for each orbit consisting only of finite circles.

We remark that $||c|^2 - |d|^2|$ is the same for every element of the coset $G_\infty gH$ but not for every element of $G_\infty gG_\infty$. Similarly, $|c|^2$ is the same for every element of $G_\infty gG_\infty$ but not for every element of $G_\infty gH$, whereas both expressions are the same for every element of $G_\infty g$, in particular for every $g^{-1}(\infty) \in P(D) \subset P(E)$.

Now consider infinite circles γ in $\Lambda(G)$. Conjugating by a Euclidean similarity if necessary we assume that γ is $\mathbb{R} \cup \{\infty\}$. As above we count G_∞ -orbits of finite circles by cosets $G_\infty \backslash G/H$. However, we want to exclude infinite circles from our summation. This means we exclude the coset of the identity and we possibly have to exclude one more coset corresponding to the other circle of infinite radius if it is in the orbit of γ . This other coset necessarily corresponds to $g_0 \in G$ with $c\bar{d} = d\bar{c}$ and $c \neq 0$. If the two infinite circles lie in the same G_∞ -orbit or in different G -orbits then there is no such $G_\infty g_0 H \in G_\infty \backslash G/H$. As above, the cosets we wish to sum over are indexed by points of $P(D)$ for a given fundamental domain D for H . Thus,

$$\sum_{g \in G_\infty \backslash G/H - \{I, g_0\}} r_2(g)^t = \sum_{g \in G_\infty \backslash G/H - \{I, g_0\}} \frac{1}{|c\bar{d} - d\bar{c}|^t} = \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{|c\bar{d} - d\bar{c}|^t}.$$

We choose D to be the union of two polygons as before. Again, the images of infinity in D are at least ε from γ . That is, for all $-d/c = g^{-1}(\infty) \in P(D)$ we have $|\text{Im}(-d/c)| \geq \varepsilon$. Now G_∞ is an extension of finite degree, say δ , of a subgroup of H . This means that D is contained in δ copies of a fundamental domain E for G_∞ . Putting these together we obtain

$$\begin{aligned} \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{|c\bar{d} - d\bar{c}|^t} &\leq \frac{1}{(2\varepsilon)^t} \sum_{g^{-1}(\infty) \in P(D)} \frac{1}{|c|^{2t}} \\ &\leq \frac{1}{(2\varepsilon)^2} \sum_{g^{-1}(\infty) \in P(E)} \frac{1}{|c|^{2t}} \\ &= \frac{\delta}{(2\varepsilon)^2} \sum_{g \in G_\infty \backslash G/G_\infty - \{I\}} \frac{1}{|c|^{2t}}. \end{aligned}$$

This proves the result for an orbit containing an infinite circle.

There can only be a finite number of orbits of circles (Ahlfors' finiteness theorem) as G is finitely generated. Thus, adding all of these series we can find C as desired. \square

We are now ready to prove the main theorem.

PROPOSITION 4. *Let G be geometrically finite and satisfy the conditions of Theorem 1. Then $e \leq d$.*

Proof. By Proposition 3, there is a constant C depending only on G so that for all $t \leq 2$,

$$\sum_{G_\infty\text{-orbits of finite circles in } \Lambda(G)} r^t \leq C \sum_{G_\infty \backslash G/G_\infty - \{I\}} |c|^{-2t}.$$

The left-hand side converges for all values of t that make the right-hand side converge. That

is, $e \leq t_0$ where t_0 is defined by

$$t_0 = \inf \left\{ t \leq 2: \sum_{G_\infty \setminus G/G_\infty - \{I\}} |c|^{-2t} < \infty \right\}.$$

This t_0 is the exponent of convergence of the Poincaré series (see [1]) and for geometrically finite groups this is d , the Hausdorff dimension of $\Lambda(G)$ (see [10], Theorem 9.3.6). Thus, $e \leq d$ as required. \square

We remark that as the Apollonian packing is the limit set of a Kleinian group this gives a new proof of the result in [5, 12].

The figures were all drawn using computer programs written by Ian Redfern, see [9].

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